International Journal of Mathematics and Physical Sciences Research ISSN 2348-5736 (Online)
Vol. 2, Issue 2, pp: (63-69), Month: October 2014 – March 2015, Available at: www.researchpublish.com

A NORM CONVERGENCE RESULT FOR THE COMPUTATION OF CORRELATION MATRIX FOR A SYMMETRIC MATRIX

Mutti Ur Rehman

Department of Mathematics in Natural, Social and Life Sciences, Gran Sasso Science Institute, 67100-L'Aquila, Italy

Abstract: **In this paper we will study the norm convergence analysis to correlation matrix for a symmetric matrix.**

Keyword: **Correlation matrix; Positive Semi definite matrix; Convex Analysis; Normal Cone; Weighted Frobenius norm; Alternating projection method; Projections; Hilbert space; Weak convergence; Strong convergence.**

I. INTRODUCTION

The correlation matrix is a symmetric positive semi definite (SPSD) matrix which occupies "1" along the diagonal. The correlation matrices occur in many areas of numerical linear algebra. It also includes linear system and the error analysis of Jacobi methods for symmetric eigenvalue problem [1]. We are interested to compute nearest correlation matrix for an arbitrary symmetric matrix $A \in R^{n\times n}$. For this, we have to compute,

- The distance $d(A) = \text{Minimize} \| A\text{-}X \|$ Where "X" is a nearest correlation matrix. (1.1)
- A matrix that should achieve this minimum distance and solve problem

In (1.1), the defined norm is the weighted version of the Frobenius norm $\| A \|_F = \Sigma_{mn} a^2_{mn}$,

$$\| A \|_{M} = \| M^{1/2} A M^{1/2} \|_F \tag{1.2}$$

In (1.2), the fixed matrix "M" is a symmetric positive definite (SPD) matrix.

We are looking for matrix in the intersection of two closed convex sets "F" and "G" which is closest to matrix "A" in weighted Frobenius norm. The close convex sets "F" and "G" are defined as following.

$$F = \{ B = B^T \in R^{n\times n} : B \geq 0 \} \text{ and } G = \{ B = B^T \in R^{n\times n} : b_{ii} = 1, i=1:n\}$$

For matrix B, $B \geq 0$ or $B \leq 0$ means that B is positive semi-definite (PSD) or negative semi definite. The minimum problem (1.1) defined as above is achieved and is unique [2]. In section II, we give the solution to problem (1.1) for M-norm. We also compute solution to problem (1.1) by using modified alternating projection method (MAPM). We project iteratively by repeating projection onto close convex sets. In section III, we study the norm convergence for the correlation matrix to given symmetric matrix.

II. THEORY

The development in this section is inspired by Glunt et al [3]. This is about treatment of the nearest Euclidean distance matrix problem.

Theorem 2.1. The correlation matrix "X" solves problem (1.1) if and only if

$$X = A + M^{-1} (SDS^T + \text{diag} (\alpha_i)) M^{-1} \tag{2.1}$$

In (2.1), $S \in R^{n\times p}$, which possesses orthonormal columns that spans null(X), $D = \text{diag}(d_i) \geq 0$ and "α_i" be an arbitrary.

Proof. In (1.2), we work with M-norm; we define an inner-product on $R^{n\times n}$ which induces M-norm.

$$\langle A , E \rangle = \text{Trace}\,(A^T MEM)$$

For convex set $K \epsilon R^{n\times n}$, the normal cone of "K" at $Z \epsilon K$ is defined as,

$$N_K(E) = \{ B = B^T \epsilon R^{n\times n} : \langle Z - E , Y \rangle \leq 0 \text{ for all } Z \epsilon K \} \quad (2.2)$$

$$= \{ B = B^T \epsilon R^{n\times n} : \langle B, E \rangle = \text{Max} \langle B, Z \rangle \text{ for all } Z \epsilon K \} \quad (2.3)$$

The solution "X" to (1.1) is characterized by the inequality [2].

$$\langle Z - X , A - X \rangle \leq 0 \text{ for all } Z \epsilon (F \cap G) \quad (2.4)$$

The above condition can be written as, $A - X \epsilon N_{(F\cap G)}(X)$ (2.5)

Which implies that,[4]. $A\text{-}X \epsilon N_F(x) + N_G(X)$

Now, we determine $N_F(x)$ and $N_G(X)$.

Lemma 2.1. For $A \epsilon G$, we have,

$$N_G(A) = \{ M^{-1}\text{diag}(\alpha_i) M^{-1} : \alpha_i \text{ is an arbitrary}\} \quad (2.6)$$

Proof. From (2.3), we have, $N_G(A) = \{ B = B^T \epsilon R^{n\times n} : \langle B, A \rangle = \text{Max} \langle B, Z \rangle \text{ for all } Z \epsilon G\}$

The constraint to above is written as, $\Sigma_{mn} \tilde{b}_{mn} a_{mn} = \text{Max}\, \Sigma_{mn} \tilde{b}_{mn} z_{mn}$,where $\tilde{B} = MBM$ and $\tilde{B}$ is a diagonal matrix, which shows that $B = M^{-1}\text{diag}(\alpha_i) M^{-1}$ where $\tilde{B} = \text{diag}(\alpha_i)$, which gives prove.

Lemma 2.2. For $A \epsilon F$, we have, That is, $N_F(A) = \{ B = B^T \epsilon R^{n\times n} : \langle B , A \rangle = 0, B \leq 0 \}$

This result is characterized by [5].

Proof. From (2.3), we have, $N_G(A) = \{ B = B^T \epsilon R^{n\times n} : \langle B , A \rangle = \text{Max} \langle B, Z \rangle \text{ for all } Z \epsilon G \}$

We take spectral decomposition of $Z \epsilon G$, as $Z = RDR^T$, where "R" is an orthogonal matrix that is $RR^T = R^T R = I$ and $D = \text{diag}(\lambda_i) \geq 0$. Then, with $P = R^T MBMR$.

So, $\langle B , A \rangle = \text{Max} \langle B , Z \rangle$ for all $Z \epsilon G$

$$= \text{Max} \langle B , RDR^T \rangle \quad \text{where } D \geq 0, \; R^T R = RR^T = I$$

This implies that,

$$= \text{Max} \langle R^T MBMR, D \rangle \quad \text{where } D \geq 0, \; R^T R = RR^T = I$$

$$= \text{Max} \langle P , D \rangle \quad \text{where } D \geq 0, \; R^T R = RR^T = I$$

$$= \text{Max}\, \Sigma_i \lambda_i p_{ii} \quad \text{where } D \geq 0, \; R^T R = RR^T = I$$

So, $\langle B , A \rangle = 0$ if $B \leq 0$, which gives prove.

For maximum condition equality holds for B so that $\langle B , A \rangle = 0$ if $B \leq 0$.

Corollary 2.3. For $A \epsilon G$, we have,

$N_G(A) = \{ B : MBM = -SDS^T$, where $D = \text{diag}(d_i) \geq 0$, $S \epsilon R^{n\times P}$ has orthogonal columns that spans null(A) $\}$. Therefore, condition (2.5) gives proof of theorem 2.1 on applying lemma 2.1 and corollary (2.3).

Modified Alternating Projection Method (MAPM)

First we consider how to project onto closed convex sets "F" and "G". We write "σ_F" and "σ_G" as the projections onto "F" and "G" respectively.

Theorem 3.1 For M-norm,

$\sigma_G(A) = A - M^{-1}\text{diag}(\alpha_i) M^{-1}$, Where $\alpha_i = [\alpha_1, \ldots, \alpha_n]^T$ be the solution of linear system given (3.1)

$$(M^{-1} o M^{-1}) \alpha = \text{diag}(A - I) \quad (3.1)$$

In (3.1), "o" stands for element wise product that is $AoB = (a_{mn} b_{mn})$.

Proof. The projection $X = б_G(A)$ which is characterized by the condition $A - X \epsilon N_G(X)$, which, by lemma 2.1 can be written as following,

$$A - X = M^{-1} \operatorname{diag}(\alpha_i) M^{-1}$$

Now, by equating the diagonal elements and writing $M^{-1} = (m_{ij})$, we have,

$\Sigma_j (m^2)_{ij} \alpha_{ij} = a_{ii} - 1$, these equations form the linear system described in (3.1). We obtain a unique solution for (3.1) due to fact "M" to be a positive definite matrix so is (M^{-1} o M^{-1}).

Theorem 3.2 For M-norm,

$$б_F(A) = M^{-1/2}((M^{1/2} A M^{1/2})_+) M^{-1/2} \quad (3.2)$$

Furthermore we also have,diag (A) ≤ diag ($б_F(A)$) Proof. Before it to prove this theorem, we use some notation. For $A \epsilon R^{n\times n}$, we write the spectral decomposition of A as, $A = QDQ^T$, where "Q" is an orthogonal matrix that is $QQ^T = Q^TQ = I$, where $D = \operatorname{diag}(\lambda_i)$. We splits the matrix "A" as, $A= A_+ + A_-$ where $A_+ = Q\operatorname{diag}(\max(\lambda_i,0))Q^T$ and similarly $A_-= Q\operatorname{diag}(\min(\lambda_i,0))Q^T$ and $A_+A_- = A_-A_+=0$.

The projection $X = б_F(A)$ which is characterized by the condition $A - X \epsilon N_F(X)$, which, by lemma 2.2 can be written as following,

$$A - X \leq 0 \text{ and } \operatorname{Trace}((A-X)MXM) = 0.$$

First we show that $A - X \leq 0$, for this we fallow as,

$$A - X = M^{-1/2}((M^{1/2}AM^{1/2}-(M^{1/2}AM^{1/2})_+)M^{-1/2}$$
$$= M^{-1/2}((M^{1/2}AM^{1/2})_-)M^{-1/2}$$
$$\leq 0, \text{ which is required proof.}$$

Now, we show that Trace ((A-X)MXM) = 0

$$(A-X)MXM = M^{-1/2}(M^{1/2}AM^{1/2})_-\, M^{-1/2}.\, M^{1/2}((M^{1/2}AM^{1/2})_+)M^{1/2}$$
$$= M^{-1/2}(M^{1/2}AM^{1/2})_-\,(M^{1/2}AM^{1/2})_+M^{1/2}$$
$$= 0, \text{ which is required proof.}$$

Now, we show that diag (A) ≤ diag($б_F(A)$)

$$\text{Since,} \quad (M^{1/2}A\, M^{1/2})_+ - M^{1/2}AM^{1/2} \geq 0$$

We pre- and post-multiply above with $M^{-1/2}$ and the we select only the diagonal parts. This gives us required result. Algorithm3.2.[6]:- For M-norm, the following algorithm gives us the nearest correlation matrix for given symmetric matrix $A \epsilon R^{n\times n}$.

$\Delta T_o = 0$, $Y_o= A$

For r = 1,2,3,…

$Q_r = Y_{r-1} - \Delta T_{r-1}$ % Dykstra's correction

$X_r = б_F(Q_r)$

$\Delta T_r = X_r - Q_r$

$Y_r = б_G(X_r)$

end

Both sequences X_r and Y_r converges to nearest correlation matrix as $r \rightarrow \infty$.

III. CONVERGENCE ANALYSIS

For convergence analysis, we adopt more general case. We consider that closed convex sets are in real Hilbert Space H. We let $K = \cap K_i$ for all i=1,2,…p be the intersection of closed convex sets. We define projection onto each and every closed convex set to study the convergence analysis. The problem (1.1) is similar to the following minimum problem,

$$\text{Minimize } \| y - f \| \tag{4.1}$$

In (4.1), f ϵ K, closed convex set and y be some fixed element of an inner-product space. Definition 4.1. Let H be a real Hilbert space. A sequence $\{x_n\}$ ϵ H is said converges strongly to x ϵ H if $\|x_n - x\| \to 0$. We write it as $x_n \xrightarrow{s} x$.

Definition 4.2. Let H be a real Hilbert space. A sequence $\{x_n\}$ ϵ H is said converges weakly to X ϵ H if $x_n \cdot y \to x \cdot y$ for all y ϵ H. We write it as $x_n \xrightarrow{w} x$.

Theorem 4.1. Let H be a real Hilbert space. Let y ϵ H be an arbitrary element. Let $K \neq \Phi$ be any non-empty closed convex set in H. Then, there exists a unique element "x" in K, the closed convex set, such that it solves minimum problem (4.1). The minimizing element "x" is characterized by the following inequality [2].

$$\langle y - x, x - f \rangle \geq 0 \quad \text{for all } f \in K \tag{4.2}$$

To solve (4.1), we suggest an algorithm (3.2) which needs to find out projection on each and every closed convex set $K = \cap K_i$ for all i=1,2,…p. For this, we define the projections as fallow.

For 1st cycle:

- $y_{11} = y + I_{11}$, where y_{11} is projection of y onto k_1
- $y_{12} = y_{11} + I_{12} = y + I_{11} + I_{12}$, where y_{12} is projection of y_{11} onto k_2
- $y_{13} = y_{12} + I_{13} = y + I_{11} + I_{12} + I_{13}$, where y_{13} is projection of y_{12} onto k_3
- .
- .
- .
- $y_{1p} = y_{1,p-1} + I_{1p} = y + I_{11} + I_{12} + I_{13} + \ldots + I_{1p}$, where $y_{1,p}$ is projection of $y_{1,p-1}$ onto k_p

For 2nd cycle:

- After 1st cycle, we first remove the increment I_{11} before projecting y_{1p} onto k_1, so for 2nd cycle we fallow.
- $y_{21} = y_{1p} - I_{11} + I_{21} = y + I_{21} + I_{12} + \ldots + I_{1p}$, where y_{21} is projection of $y_{1p}-I_{11}$ onto k_1
- $y_{22} = y_{21} - I_{12} + I_{22} = y + I_{21} + I_{22} + I_{13} + \ldots + I_{1p}$, where y_{22} is projection of $y_{21}-I_{12}$ onto k_2
- $y_{23} = y_{22} - I_{13} + I_{23} = y + I_{21} + I_{22} + I_{23} + I_{14} + \ldots + I_{1p}$, where y_{23} is projection of $y_{22}-I_{13}$ onto k_3
- .
- .
- .
- $y_{2p} = y_{2,p-1} - I_{1p} + I_{2p} = y + I_{21} + I_{22} + I_{23} + \ldots + I_{2p}$, where y_{2p} is projection of $y_{2,p-1}-I_{1p}$ onto k_p

By a continuous manner of above, we get the sequences $\{ y_{ni} \}$ and $\{ I_{ni} \}$ for all $1 \leq i \leq p$ and $n \geq 1$. We can accumulate above discussion for I = 2,3,… and $n \geq 1$ as following.

So,

$$\text{(a)} \quad I_{n-1,1} - I_{n1} = y_{n-1,p} - y_{n1}$$

$$\text{(b)} \quad I_{n-1,p} - I_{np} = y_{n,p-1} - y_{np} \tag{4.3}$$

In (4.3), we allow $y_{0p} = y$ and $I_{0p} = 0$ for all p. So, from this we obtained following characterization.

$$y_{np} = y + I_{n1} + \ldots + I_{np} + I_{n-1,p+1} + \ldots + I_{n-1,p} \quad \text{for all } 1 \leq i \leq p \text{ and } n \geq 1. \tag{4.4}$$

Theorem 4.2. The sequence $\{ y_{ni} \}$ converges strongly to minimizing element "x" that is,

$$\| y_{ni} - x \| \to 0 \text{ as } n \to \infty \text{ for all } 1 \leq i \leq p$$

Proof of Theorem 4.2. We start from the following Equalities,

$$\| y - x \|^2 = \| (y_{11} - x) - I_{11} \|^2$$

$$= \| y_{11} - x \|^2 - 2 \langle y_{11} - x , I_{11} \rangle + \| I_{11} \|^2$$

$$= \| y_{11} - x \|^2 + 2 \langle y_{11} - x , y - y_{11} \rangle + \| I_{11} \|^2 \quad (4.5)$$

The middle term is non-negative due to (4.2). By a similar treatment we do decomposition for $\| y_{11} - x \|$.

$$\| y_{11} - x \|^2 = \| y_{12} - x \|^2 + 2 \langle y_{12} - x , y_{11} - y_{12} \rangle + \| I_{12} \|^2$$

So, (4.5) takes the form,

$$\| y - x \|^2 = \| y_{12} - x \|^2 + 2 \langle y_{12} - x , y_{11} - y_{12} \rangle + 2\langle y_{11} - x , y - y_{11} \rangle + \| I_{11} \|^2 + \| I_{12} \|^2$$

If we continue this process, then through 1st cycle we get,

$$\| y - x \|^2 = \| y_{1p} - x \|^2 + 2 \Sigma_i \langle y_{1,i-1} - y_{1i} , y_{1i} - x \rangle + \Sigma_i \| I_{1i} \|^2 \text{ for all } 1 \le i \le p, \text{ we set } y_{10} = y. \quad (4.6)$$

In (4.6) all terms are non-negative.

Also, $\| y_{1p} - x \|^2 = \| y_{21} - x \|^2 + 2 \langle y - y_{11} , y_{11} - y_{21} \rangle + 2 \langle y_{1p} - I_{11} - y_{21} , y_{21} - x \rangle + \| I_{11} - I_{21} \|^2$

So, in view of above (4.6) gives us,

$\| y - x \|^2 = \| y_{21} - x \|^2 + 2 \langle y - y_{11} , y_{11} - y_{21} \rangle + 2 \langle y_{1p} - I_{11} - y_{21} , y_{21} - x \rangle + 2 \Sigma_{i=2:p} \langle y_{1,i-1} - y_{1i} , y_{1i} - x \rangle + \Sigma_{i=1:p} \| I_{1i} \|^2 + \| I_{11} - I_{21} \|^2$.

By continue this process to nth cycle and let i = p, we get

$$\| y - x \|^2 = \| y_{np} - x \|^2 + \Sigma_{m=1:n} \Sigma_{i=1:p} \| I_{m-1,i} - I_{mi} \|^2 + 2 \Sigma_{m=1:n-1} \Sigma_{i=1:p} \langle y_{m,i-1} - I_{m-1,i} - y_{mi} , y_{mi} - y_{m+1,i} \rangle + 2 \Sigma_{i=1:p} \langle y_{n,i-1} - I_{n-1,i} - y_{ni} , y_{ni} - x \rangle \quad (4.7)$$

In (4.7) we set, $y_{m0} = y_{m-1,p}$ for all "m" and $I_{0i} = 0$ for "i". Here also all terms are non-negative for "n".

Hence, in (4.7), we have infinite sum described as,

$$\Sigma_{m=1:\infty} \Sigma_{i=1:p} \| I_{m-1,i} - I_{mi} \|^2 < \infty \quad (4.8)$$

also clear that (4.8) along with (a) and (b) of (4.3) gives sequence of successive increments. So,

$$\| y_{11} - y_{12} \| , \| y_{12} - y_{13} \| ,\ldots, \| y_{1p} - y_{21} \| ,\ldots, \| y_{n,i-1} - y_{ni} \| \to 0$$

Hence, the sequence $\{ y_{np} \}$ converges strongly to minimizing element "x" of (4.1) if and only if the sequence $\{ y_{ni} \}$ converges strongly to minimizing element "x" for all "i".

Now, let $f \in K = \cap K_i$ for all i = 1,2,…p, for $n \ge 1$, we write as,

$$\langle y_{np} - y , y_{n1} - f \rangle = \langle I_{n1} + \ldots + I_{np} , y_{n1} - f \rangle$$

$$= \langle I_{n1} , y_{n1} - f \rangle + \ldots + \langle I_{np} , y_{n1} - f \rangle$$

$$= \langle I_{n1} , y_{n1} - y_{n1} + y_{n1} - f \rangle + \langle I_{n2} , y_{n1} - y_{n2} + y_{n2} - f \rangle + \ldots + \langle I_{np} , y_{n1} - y_{np} + y_{np} - f \rangle$$

$$= \langle I_{n2} , y_{n1} - y_{n2} \rangle + \ldots + \langle I_{np} , y_{n1} - y_{np} \rangle + [\langle I_{n1} , y_{n1} - f \rangle + \ldots + \langle I_{np} , y_{np} - f \rangle] \quad (4.9)$$

By using properties of absolute value and Cauchy-Shwarz Inequality for the expression as

$\langle I_{n2} , y_{n1} - y_{n2} \rangle + \ldots + \langle I_{np} , y_{n1} - y_{np} \rangle$ in (4.9), we get following

$$\left| \langle I_{n2} , y_{n1} - y_{n2} \rangle + \ldots + \langle I_{np} , y_{n1} - y_{np} \rangle \right| \le \| I_{n2} \| \| y_{n1} - y_{n2} \| + \ldots + \| I_{np} \| \| y_{n1} - y_{np} \| \quad (4.10)$$

(4.10) it can also be noted that,

$\| y_{n1} - y_{ni} \| = \| y_{n1} - y_{n2} + y_{n2} - y_{n3} + \ldots - y_{n,i-1} - y_{ni} \| \le \| y_{n1} - y_{n2} \| + \| y_{n2} - y_{n3} \| + \ldots + \| y_{n,p-1} - y_{np} \| = S_n$ (say)

In view of above (4.10) implies that,

$$\left| \langle I_{n2} , y_{n1} - y_{n2} \rangle + \ldots + \langle I_{np} , y_{n1} - y_{np} \rangle \right| \le \Sigma_{i=2:p} \| I_{ni} \| S_n$$

Also, $\| I_{ni} \| = \| \Sigma_{m=1:n}(I_{mi} - I_{m-1,i}) \| \le \Sigma_{m=1:n} \| I_{mi} - I_{m-1,i} \|$, which implies that (4.10) is bounded above by $\Sigma_{m=1:n} \Sigma_{i=2:p} \| I_{mi} - I_{m-1,i} \| S_n$

Now, by using (b) of (4.3) and S_n define as above as, we get the following,

$$\Sigma_{m=1:n} \Sigma_{i=2:p} \| I_{mi} - I_{m-1,I} \| S_n = \Sigma_{m=1:n} \Sigma_{i=2:p} \| y_{m,i-1} - y_{mi} \| S_n = \Sigma_{m=1:n} S_m S_n$$

We are in position to show that $\Sigma_{n=1:\infty} S^2_n < \infty$. For this we let $w_i = ||I_{n-1,i} - I_{n,i}||$ for all $i = 2,3,\ldots,p$. Then,

$$S^2_n = (\Sigma_{i=2:p} w_i)^2 = \Sigma_{i=2:p} w^2_i + 2 \Sigma_{i<j} w_i w_j$$, where for last sum we have,

$(q-1)(q-2)/2$ terms. Since $(w_i - w_j)^2 \geq 0$, this implies that $2 w_i w_j \leq w^2_i + w^2_j \leq \Sigma_{i=2:p} w^2_i$, which gives us,

$$\Sigma_{n=1:\infty} S^2_n \leq [(q-1)(q-2)/2 + 1] \Sigma_{n=1:\infty}\Sigma_{i=2:p} || I_{n-1,i} - I_{ni} ||^2$$

From (4.8), it's clear that right hand side of above is finite. So, $\Sigma_{n=1:\infty} S^2_n < \infty$.

Lemma 4.1.[7] Let $\{ S_n \}$ be a sequence of non-negative real numbers with $\Sigma_{n=1:\infty} S^2_n < \infty$, then there exists $\{ S_{nj} \}$, a subsequence such that $\Sigma_{m=1:nj} S_m S_{nj} \rightarrow 0$ as $j \rightarrow \infty$

Since, it's clear that (4.10) is bounded above by $\Sigma_{m=1:n} S_m S_n$ for all "n". From Lemma (4.1), we observe that (4.10) tends to zero for large "j" that is $j \rightarrow \infty$ for $n \rightarrow n_j$.

For any $f \in K$ (4.9) implies that,

$$\lim_j \langle y_{nj,p} - y , y_{nj,1} - f \rangle \leq 0 \text{ for all } f \in K \qquad (4.11)$$

This implies that, $\lim_j \langle y_{nj,p} - y , y_{nj,1} - y_{nj,p} + y_{nj,p} - f \rangle \leq 0$ for all $f \in K$

Which further can be written as,

$$\{ \langle y_{nj,p} - y , y_{nj,1} - y_{nj,p} \rangle + \langle y_{nj,p} - y , y_{nj,p} - f \rangle\} \leq 0 \text{ for all } f \in K.$$

We use Cauchy-Shwarz Inequality for $\langle y_{nj,p} - y , y_{nj,1} - y_{nj,p} \rangle$ to get,

$$| \langle y_{nj,p} - y , y_{nj,1} - y_{nj,p} \rangle | \leq || y_{nj,p} - y || \, || y_{nj,1} - y_{nj,p} ||$$

From (4.7), we have $|| y_{nj,p} - x||$ is uniformly bounded in "j" while discussion from (4.8), we have,

$$|| y_{nj,1} - y_{nj,p} || \rightarrow 0 \text{ as } j \rightarrow \infty$$

Therefore (4.11) implies that, $\lim_j \langle y_{nj,p} - y , y_{nj,p} - f \rangle \leq 0$ for all $f \in K$ (4.12)

So, there should be a subsequence in $\{ y_{nj,p} \}$ which converges weakly to $u \in H$ that is

$$y_{nj,p} \xrightarrow{W} u \text{ as } j \rightarrow \infty \qquad (4.13)$$

Since, $|| y_{nj,p} ||$ is also bounded, we let a subsequence $\{ n_j \}$ such that, $|| y_{nj,p} || \rightarrow v \geq 0$ as $j \rightarrow \infty$ (4.13)

We use the result from Balakrishnan [7], which says that,

If the $\{ x_n \}$ converges weakly to "x" that is, $\{ x_n \} \xrightarrow{W} x$ and $|| x_n || \rightarrow x$, then $|| x || \leq t$.

So, by using above result, (4.13) and (4.14) implies that $|| u || \leq v$.

$$0 \geq \lim_j \langle y_{nj,p} - y , y_{nj,p} - f \rangle$$

This implies that,

$$= v^2 - \langle u , f \rangle - \langle u , y \rangle + \langle y , f \rangle$$
$$\geq || u ||^2 - \langle u , f \rangle - \langle u , y \rangle + \langle y , f \rangle$$
$$\geq \langle u - y , u - f \rangle$$

This implies that, $\langle y - u , u - f \rangle \leq 0$ for all $f \in K$.

Now, next we show that $u = x \in K \subseteq H$, for this we verify that $u \in K$. We fallow the result from Balakrishnan [6].

The sequence $\{ y_k \}$ such that $\frac{1}{r} [y_{nj1,i} + \ldots + y_{njr,i}] \xrightarrow{S} u$ as $r \rightarrow \infty$. Here sum in square bracket is the convex combination of elements in convex sets "K_i" for all "r" and lies in "K_i" for all "r". This implies that $u \in K_i$ for all "i" because "K_i" be closed convex set for all "i". Therefore, $u \in K$ and give $u = x \in K \subseteq H$. By taking $f = u$, from (4.12) we have the following,

$$\lim_j \langle y_{nj,p} - y , y_{nj,p} - u \rangle = v^2 - || u ||^2 \leq 0 \qquad (4.14)$$

This implies that, $v = \| u \|$

Theorem 4.3.[7]:- Consider $x_n \xrightarrow{W} x$ and $\| x_n \| \to \| x \|$,Then $x_n \xrightarrow{S} x$.

In view of theorem (4.3) we have, $y_{nj,p} \xrightarrow{S} u = x$. So the sub-sequence $\{ y_{np} \} \xrightarrow{S} x$.

Now, we adopt a similar treatment that was for derivation of (4.7) and get the following result,

$$\| y_{nj,p} - x \|^2 = \| y_{nj+b,p} - x \|^2 + \Sigma_{i=1:p} \Sigma_{q=1:b} \| I_{nj+q-1,i} - I_{nj+q,I} \|^2 + 2 \Sigma_{i=1:p} \Sigma_{q=1:b} \langle I_{nj+q-1,i} - I_{nj+q,i} , y_{nj+q,i} - x \rangle \quad (4.15)$$

We split up last double sum into difference of two double sums and first re indexed to set equation (4.15) as,

$$= 2 \Sigma_{i=1:p} \Sigma_{q=0:b-1} \langle I_{nj+q,i} , y_{nj+q+1,i} - x \rangle - 2 \Sigma_{i=1:p} \Sigma_{q=1:b} \langle I_{nj+q,i} , y_{nj+q,i} - x \rangle \quad (4.16)$$

$$= 2 \Sigma_{i=1:p} \langle I_{nj,i} , y_{nj+1,i} - x \rangle + 2 \Sigma_{i=1:p} \langle -I_{nj+b,i} , y_{nj+b,i} - x \rangle + 2 \Sigma_{i=1:p} \Sigma_{q=1:b-1} \langle I_{nj+q,i} , y_{nj+q+1,i} - y_{nj+q,I} \rangle \quad (4.17)$$

In equation (4.17), we have

$$2 \Sigma_{i=1:p} \langle I_{nj,i} , y_{nj+1,i} - x \rangle = 2 \Sigma_{i=1:p} \langle I_{nj,i} , y_{nj+1,i} - y_{nj,I} + y_{nj,i} - x \rangle$$

$$= 2 \Sigma_{i=1:p} \langle I_{nj,i} , y_{nj+1,i} - y_{nj,i} \rangle + 2 \Sigma_{i=1:p} \langle I_{nj,i} , y_{nj,i} - x \rangle \quad (4.18)$$

In (4.18) each term is non-negative due to (4.2). From above discussion it's clear that

$$\langle y_{nj,p} - y , y_{nj,1} - x \rangle \to 0 \text{ as } j \to \infty$$

This shows that (4.9) tends to zero for $n \to n_j$ and $f = x$ which further clear the idea that as "j" increases then we get as $2 \Sigma_{i=1:p} \langle I_{nj,i} , y_{nj,i} - x \rangle$ in equation (4.18) vanishes too. So, from this we have

$\| y_{nj,p} - x \|^2$ consists of non-negative terms and for $y \to \infty$, the sum goes to zero. That is,

$$\| y_{nj+r} - x \|^2 \to 0 \text{ as } j \to \infty \quad \text{for all } r \geq 0$$

Since, $\| y_{nj,p} - x \|^2 \to 0$ which gives the required result for convergence analysis of theorem (4.2).

REFERENCES

[1] DAVIES, P. I. & HIGHAM, N. J. (2000) Numerically stable generation of correlation matrices and their factors. BIT, 40, 640-651.

[2] LUENBERGER, D. G (1969, p.69) Optimization by Vector Space Methods. New York: Wiley.

[3] GLUNT, W., HAYDEN, T. L., HONG, S. & WELLS, J. (1990) An alternating projection algorithm for computing the nearest Euclidean distance matrix. SIAM J. Matrix Anal. Appl., 11, 589-600.

[4] ROCKAFELLAR, T. R. (1970) Convex Analysis. Princeton, NJ: Princeton University Press.

[5] FLETCHER, R, (1985) Semi-definite matrix constraints in optimization. SIAM J. Control and Optimization, 23, 393-513.

[6] NICHOLAS J, HIGHAM, Computing the Nearest Correlation Matrix, A problem from Finance. IMA Journal Numerical Analysis (2002) 22, 329-343.

[7] DYKSTRA, R. L. (1983). "An algorithm for restricted least squares regression." J. Amer. Statist. Assoc. 78: 837-842.

[8] BALAKRISHNIAN, A.V. (1971). Introduction to Optimization Theory in a Hilbert Space. Springer-Verlag, Berlin.